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Effect of CP-violation on the sphaleron rate

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Abstract

We calculate the effect of two CP-violating (dimension-eight) operators of the SU(2)-Higgs model on the motion along a particular path from the vacuum to the sphaleron. It turns out that CP-violation may introduce a difference between the sphaleron rate towards larger Chern-Simons number and the rate towards smaller Chern-Simons number. Such a difference induces a non-zero baryon-number without a first order phase transition.

The matter anti-matter asymmetry of the present universe is an important cosmological observation. It can be given a quantitative meaning by considering the ratio of the net baryon-number density and photon density [1]

$$\frac{\Delta_B}{n_\gamma} \sim 10^{-10}, \tag{1}$$

with $\Delta_B = n_B - n_{\bar{B}}$ the difference between the baryon-number density and the anti-baryon-number density. This ratio is constant under the expansion of the universe. Contrary to its superficial appearance the baryon-number excess (1) is actually very large to be explained by the standard model. The problem is to explain the generation of this amount of baryons, when the universe started out from a state with baryon-number equal to zero (see for recent reviews e.g. [3, 4]). In 1967 Sakharov [2] was the first to address this problem and he noted that there are three prerequisites, namely
1) non-conservation of baryon-number,
2) C- and CP-violation,
3) departure from equilibrium.
In relation to the first requirement, it was discovered by 't Hooft [5] that in the standard model the baryon number is not conserved, as a consequence of the anomaly equation

$$\partial_\mu j^\mu_B = \frac{3g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu a}, \tag{2}$$

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and transitions between different vacua that are labeled by the Chern-Simons number. A transition from one (classical) vacuum to the next in the positive (negative) Chern-Simons direction yields a change in baryon-number of $+3(-3)$. These transitions are possible because of quantum tunneling through the barrier separating the different vacua, but at zero temperature they are very much suppressed. At high temperatures the system can go over the barrier due to thermal fluctuations, because of that the rate of baryon-number changing transitions can be quite large. The transition rate is proportional to the Boltzmann factor: $\Gamma_{\rm sph} \sim \exp -E_{\rm sph}/T$ [6, 7, 8], with the energy of the sphaleron $E_{\rm sph} = \text{number} \times 4\pi v/g$, where v is the expectation value of the Higgs field at temperature T. It is the sphaleron energy that occurs in the Boltzmann factor since the sphaleron is the minimal energy configuration at the barrier.

In the standard model CP-violation occurs in the CKM-matrix; however it is too small to explain the observed number of baryons. This is an indication for physics beyond the standard model with extra CP-violation (such as the two Higgs doublet model or the minimal supersymmetric standard model).

Also it has been established that in the standard model there is no (strong) first-order electroweak phase transition [9]. This also has been taken as an indication for new physics that should provide a strong enough departure from equilibrium. The requirement for a strong first-order electroweak phase transition has been used to constrain parameters of extensions of the standard model [10]. Also new mechanisms for a departure from equilibrium at the electroweak scale have been considered, see for example [11].

In this letter we want to point out the possibility that in a model with sufficient CP-violation, the baryon-number expectation value is non-zero when the system is in kinetic equilibrium, but sectors with a different baryon-number are not in equilibrium. A non-zero expectation value of the B-number can occur when the rate of sphaleron transitions to the vacuum with a larger Chern-Simons number, $\Gamma^{\uparrow}$, is different from the rate of transitions to the vacuum with a smaller Chern-Simons number $\Gamma^{\downarrow}$ (an example of such a difference out of equilibrium is discussed in [12]). If this is the case, an initial state with zero baryons will evolve into a state with non-zero baryon-number.

The question is how CP-violating interactions may induce such a difference in rates. We discuss this for a specific CP-violating action

$$S_{CP} = \int d^4x \frac{1}{M^4}\left[\delta^1_{CP}(D_\rho\phi)^\dagger(D^\rho\phi) - \delta^2_{CP}\frac{1}{4}F^a_{\rho\sigma}F^{\rho\sigma a}\right]\frac{3g^2}{32\pi^2}F^b_{\mu\nu}\tilde{F}^{\mu\nu b}. \qquad (3)$$

This action may be thought of to come from integrating out new (CP-violating) physics at the mass-scale M (which may be temperature-dependent, for instance through the temperature-dependence of v). The operators in (3) are the lowest dimensional CP-odd operators in the SU(2)-Higgs sector that will contribute to the baryon number expectation value. We will see that the dimension-six operator $\phi^\dagger\phi F\tilde{F}$ does not give a contribution.

To study the effect of CP-violation on sphaleron transitions, we consider the motion along a specific path starting at the vacuum and ending at the

sphaleron at $N_{CS} = +1/2$. For simplicity we use the path introduced in [13]. This path is not the minimal energy path, which was constructed in [14]. But we expect that the precise path will not be important for the following rather general arguments and that the final result is sufficient as an order of magnitude estimate. We parameterize the path by the (time-dependent) angle $\Theta \in [0, \frac{1}{2}\pi]$, and use the following Ansatz for the fields (in the radial gauge)

$$
\begin{aligned}
A_\mu^a \sigma^a &= \frac{-2i}{g} f(r)[\partial_\mu U(\Theta)]U^{-1}(\Theta), &(4)\\
\phi &= \frac{1}{2}\sqrt{2} v h(r) U(\Theta) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, &(5)
\end{aligned}
$$

with the Θ-dependent SU(2)-matrix

$$
U(\Theta) = \frac{1}{r} \begin{pmatrix} z & x+iy \\ -x+iy & z \end{pmatrix} \sin\Theta + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \cos\Theta. \qquad (6)
$$

This parameterization is a non-static generalization of the fields considered in [13, 6] (with the identification $\mu = \Theta$). This particular generalization is convenient, since the field strength vanishes at infinity for the asymptotic boundary condition $r \to \infty$ $f(r) \to 1$.

We insert the fields (4) and (5) in the SU(2)-Higgs action and the CP-violating action (3) and find

$$
\begin{aligned}
S &= \int dt \left[\frac{4\pi v^2}{(gv)^3} \left(a_1 + a_2 \sin^2\Theta\right) \dot{\Theta}^2 - \frac{4\pi v}{g} \left(a_3 \sin^2\Theta + a_4 \sin^4\Theta\right) \right], &(7)\\
S_{CP} &= \frac{4\pi v^2}{M^4} \int dt \left(b_1 \delta_{CP}^1 + b_2 \delta_{CP}^2 + b_3 \delta_{CP}^2 \sin^2\Theta\right) \dot{\Theta}^3 \sin^2\Theta. &(8)
\end{aligned}
$$

In the CP-violating action we have neglected total time-derivatives. Had we included the dimension-six operator $\phi^\dagger \phi F \tilde{F}$ it would only have given a total time derivative. The coefficients a_1, a_2, a_3, a_4, b_1, b_2, and b_3 stand for integrals involving $f(r)$, $h(r)$, $\partial_r f(r)$, and $\partial_r h(r)$. We use Ansatz b of [6] for the functions $f(r)$ and $h(r)$; then the parameters only depend on the ratio λ/g^2, with λ the Higgs self-coupling. We take $\lambda = g^2$, for $g \approx 0.65$ this sets the (zero temperature) Higgs mass at 230 GeV, and find for the coefficients the numerical values

$$
\begin{aligned}
a_1 &= 2.51,\\
a_2 &= 1.35,\\
a_3 &= 1.58,\\
a_4 &= 0.53,\\
b_1 &= 0.14,\\
b_2 &= 0.096,\\
b_3 &= 0.23. &(9)
\end{aligned}
$$

The CP-odd action (8) introduces a velocity-dependent force in the equations of motion. For the moment we ignore the $\sin\Theta$-dependence in (8). Then the force points in the direction of motion when the system moves from the vacuum towards the sphaleron at $N_{CS} = +\frac{1}{2}$, whereas the force is opposite to the direction of motion when the motion is towards the sphaleron at $N_{CS} = -\frac{1}{2}$. As a consequence, the system will find it easier to cross the barrier to the right than to the left. Therefore we expect that the probability of crossing the barrier to the right, $P^{\uparrow}$, is larger than the probability of crossing the barrier to the left, $P^{\downarrow}$.

To obtain a quantitative estimate for the effect of the CP-odd terms on the motion over the barrier, we consider the shift in the energy caused by the extra CP-violating terms (8)

$$E_{CP}(\Theta,\dot{\Theta}) = \frac{8\pi v^2}{M^4}\left(b_1\delta^1_{CP} + b_2\delta^2_{CP} + b_3\delta^2_{CP}\sin^2\Theta\right)\dot{\Theta}^3\sin^2\Theta. \tag{10}$$

Especially the typical energy shift at the sphaleron configuration is important. To calculate this energy shift, we need the typical velocity $\dot{\Theta}$. To zeroth-order in δ^1_{CP} and δ^2_{CP} the velocity is Gaussian distributed at the sphaleron and we find

$$\langle\dot{\Theta}^2\delta(\Theta - \frac{1}{2}\pi)\rangle = \frac{(gv)^3 T}{4\pi v^2(a_1+a_2)}, \tag{11}$$

where the δ-function enforces that the average over the velocity is taken at the sphaleron configuration. With this estimate for the velocity we find for the typical energy shift

$$\delta E_{\rm sph} = \frac{1}{\sqrt{\pi}vM^4}\left(b_1\delta^1_{CP} + b_2\delta^2_{CP} + b_3\delta^2_{CP}\right)\left[\frac{(gv)^3 T}{(a_1+a_2)}\right]^{\frac{3}{2}}, \tag{12}$$

which provides a quantitative measure for the amount of CP-violation.

As an estimate for $P^{\uparrow}$ we may take the probability that a configuration at the barrier moves in the positive Chern-Simons direction

$$P^{\uparrow} = \langle\delta(\Theta - \frac{1}{2}\pi)H(\dot{\Theta})\rangle, \tag{13}$$

where $H(\dot{\Theta})$ is the Heaviside function. In a similar manner $P^{\downarrow}$ can be calculated. We get

$$P^{\uparrow\,(\downarrow)} = \frac{1}{2} + (-)0.80\,\beta\delta E_{\rm sph}. \tag{14}$$

We are interested in the case that $T << E_{\rm sph}$. Then the time that is spend rolling down is much smaller than the time spend around the vacuum in between two barriers. In this case we may neglect the effect of noise (from the other degrees of freedom that were not taken into account in our Θ-analysis) during the motion from one vacuum to the next. And the difference in rates towards negative or positive Chern-Simons number is approximately the difference in the probabilities (14)

$$\Gamma^{\uparrow\,(\downarrow)} = \Gamma_{\rm sph}\left[1 + (-)\,c\beta\delta E_{\rm sph}\right], \tag{15}$$

where c is a coefficient of order one. In the estimate for the upward and downward sphaleron rates (15) in the presence of CP-violating interactions (3) an uncertainty arises from the path that we have chosen, because the fields (4) and (5) do not satisfy the (SU(2)-Higgs) equations of motion. However, for $\Theta = \frac{1}{2}\pi, \dot{\Theta} = 0$ these fields provide a very good approximation to the solution of the (static) field equations [6]. Hence, we expect that close to the sphaleron and for small velocities $\dot{\Theta} << gv$, the estimates (12) and (15) provide a reasonable approximation. In any case, the parametric dependence on g, v, M, and T should be correct.

The difference in rates (15) has been obtained by treating a single transition as a classical motion over the barrier. The dynamics at a larger scale involving more transitions is different, namely that of a random walk with different probabilities of moving left or right. This difference in rates or probabilities implies that the expectation value of the Chern-Simons number grows linearly in time

$$\langle N_{\mathrm{CS}}(t) - N_{\mathrm{CS}}(t_{\mathrm{in}})\rangle = V\left(\Gamma^{\uparrow} - \Gamma^{\downarrow}\right)(t - t_{\mathrm{in}}), \tag{16}$$

with V the volume. The brackets denote a classical average over initial conditions with a normalizable probability distribution. Note that (16) is CPT-invariant.

When we include the baryons into the system there is no infinite growth of the baryon-number expectation value, because there is an effective potential of the baryon-number that opposes the effect. For small baryon-number densities the potential is quadratic: $V_{\mathrm{eff}}(\Delta_B) \sim \Delta_B^2$. Therefore also a non-zero baryon-number will induce a difference in rates [8, 15]. Combining the effect of a non-zero baryon-number density and CP-violation to first order, we find for the rates

$$\Gamma^{\uparrow(\downarrow)}(\Delta_B) = \Gamma_{\mathrm{sph}}\left[1 - (+)\, 0.80\frac{\Delta_B}{n_\gamma} + (-)\, c\frac{\delta E_{\mathrm{sph}}}{T}\right], \tag{17}$$

where $n_\gamma = 0.24\, T^3$ is the photon density. The rate equation is

$$\frac{\mathrm{d}\Delta_B}{\mathrm{d}t} = 3\left[\Gamma^{\uparrow}(\Delta_B) - \Gamma^{\downarrow}(\Delta_B)\right]. \tag{18}$$

From the rate equation we find the stationary (and stable) solution

$$\frac{\Delta_B}{n_\gamma} = 1.25c\frac{\delta E_{\mathrm{sph}}}{T}. \tag{19}$$

Also of interest is the width in the distribution of baryon-number densities. When the mean value and the width are small compared to T^3 the width of the distribution increases through diffusion. Hence the width is of order $[\Gamma_{\mathrm{sph}}(t - t_{\mathrm{in}})/V]^{1/2}$ at time t. We see that it is suppressed by the volume of the system. We can also determine the time-scale of equilibration of the system. We expect that the system starts to equilibrate when the distribution reaches the upperbound provided by energy conservation of the classical subsystem. The asymmetry (19) will then decrease and eventually vanish, as it should in equilibrium (see e.g. [4]). The energy of the classical system is of order VT^4. We

find that the time where the width is of order T^3 determines the time-scale of equilibration. Using the estimate for the width given above, we find that the equilibration time diverges in the infinite volume limit.

From the results obtained above a scenario for baryon-number generation in the universe may be constructed. Consider the situation that the universe at some time before the electroweak phase transition (or cross-over) is in a state with baryon-number density equal to zero. This initial condition may be provided by inflation. Since the equilibration time is extremely long the broken phase will then be entered with zero baryon-number density. In the broken phase the value (19) will be reached in a relatively short time. As the universe expands and the temperature decreases, the baryon-photon ratio (19) decreases as $T^{\frac{1}{2}}$. Also the rate of the sphaleron transitions decreases. Below the temperature $T^* \approx v(T^*) \approx 100$ GeV the baryon-number is frozen out [3]. From (9), (12), and (19) we obtain for the resulting baryon-number

$$\left.\frac{\Delta_B}{n_\gamma}\right|_{\rm now} = \left(7\,\delta^1_{CP} + 16\,\delta^2_{CP}\right) \times 10^{-5} \left(\frac{100\ {\rm GeV}}{M}\right)^4, \tag{20}$$

where we have used $c = 1$. Also we have included the factor 0.037 to account for the changes in the ratio due to changes in the number of relativistic particle species when the universe was cooling down. In the standard model the magnitude of δ^1_{CP}, δ^2_{CP} is too small (about 10^{-20}) to explain the observed matter anti-matter asymmetry (1). However, for extensions of the standard model $\delta^1_{CP}, \delta^2_{CP}$ can be as large as 10^{-3} and we see that (20) may explain the observed baryon-number excess (1), without introducing a (strong) first-order phase-transition.

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